

Continuum Nonsimple Loops and 2D Critical Percolation

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Substantial progress has been made in recent years on the 2D critical percolation scaling limit and its conformal invariance properties. In particular, chordal SLE_6 (the Stochastic Loewner Evolution with parameter $\kappa = 6$) was, in the work of Schramm and of Smirnov, identified as the scaling limit of the critical percolation “exploration process.” In this paper we use that and other results to construct what we argue is the *full* scaling limit of the collection of *all* closed contours surrounding the critical percolation clusters on the 2D triangular lattice. This random process or gas of continuum nonsimple loops in \mathbb{R}^2 is constructed inductively by repeated use of chordal SLE_6 . These loops do not cross but do touch each other—indeed, any two loops are connected by a finite “path” of touching loops.

KEY WORDS: Scaling limit; percolation; SLE; continuum loops; nonsimple loops; triangular lattice; conformal invariance.

1. INTRODUCTION

We are honored to participate in the special issue of JSP dedicated to Elliott Lieb on his 70th birthday. He is an inspiration both to those starting on their careers and to those who believe that there is no age cutoff for outstanding research.

Percolation is a model with a wide range of applications and, especially in two dimensions, a well developed theory (see, for example, refs. 1 and 2). It has been used as a proving ground for developing tools that can be applied to more complex systems, and is of great interest in its own

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right, as it is perhaps the simplest (non-mean-field) model displaying a phase transition with features such as scaling and universality at criticality.

In the critical case, the fractal and conformally invariant nature of (the scaling limit of) large percolation clusters has attracted much attention and is of interest for both intrinsic reasons and as a paradigm for the study of other systems.

The ground-breaking work of Schramm⁽³⁾ and Smirnov⁽⁴⁾ has elucidated much about the nature of the scaling limit of the cluster boundaries or contours in terms of SLE_6 , the Stochastic Loewner Evolution with parameter $\kappa = 6$. Important and related work by Lawler–Schramm–Werner^(5–11) and Smirnov–Werner⁽¹²⁾ has yielded a multitude of results on exponents, conformal invariance and other properties of critical percolation and other two-dimensional processes (excellent reviews are given in refs. 13 and 14). To extend the work of Schramm and Smirnov, in the spirit of Aizenman⁽¹⁵⁾ and Aizenman–Burchard⁽¹⁶⁾ (see also ref. 17), it is natural to treat the scaling limit for the *set of all contours*, as was considered in ref. 18 (see also Theorem 2.1 of ref. 9). But to our knowledge, no complete description of that *full* scaling limit and its relation to SLE_6 has been proposed, although very interesting ideas do appear in ref. 18 (see Theorem 4 and Section 3.3 there) and ref. 9 (see Theorem 2.1 there). In ref. 19, a certain critical dependent percolation model on the triangular (or hexagonal) lattice was proved to have the same scaling limit for all of its contours as in the independent triangular case, even though the full scaling limit itself had not been identified.

In this paper, we present an inductive construction using chordal SLE_6 , which results in a random process (or gas) of continuum nonsimple loops in the plane. The construction is given in Section 2 and then a number of features that we argue are valid for this process are presented in Section 3. Chief among these features is that this process of continuum nonsimple loops is indeed the scaling limit (without need for subsequences) of the set of all boundary contours for critical site percolation on the triangular lattice. A technical property, which will be used to argue for the scaling limit feature, is that various ways of organizing the construction give the same limiting distribution. Sketches of the main arguments for the claimed features, using refs. 3 and 4 and other work, are provided in Section 4. A paper by the authors with detailed proofs is in preparation.

Another important property of the loop process is conformal invariance; we do not discuss that explicitly since it is essentially the same as in the conformal invariance results of Lawler–Schramm–Werner^(5,6) and Smirnov⁽⁴⁾ (see also refs. 13 and 14). We remark that in particular, the distribution of the loop process on the entire plane will be scale and inversion invariant, in addition to being translation and rotation invariant.

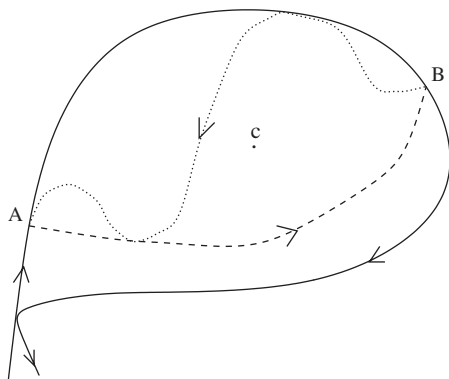


Fig. 1. Construction of a continuum loop around c in three steps.

As a preview of the way in which a single one of our continuum loops is constructed, see the (very schematic) Fig. 1, in which *three* chordal SLE_6 processes are used to yield a single loop surrounding a point c in the plane: The process γ_1 (solid curve in the figure), when it first traps c provides a domain D_1 for the second process γ_2 . A domain D_2 for the third process γ_3 (dotted curve) is provided when γ_2 makes an *excursion* (dashed curve) from A to B , two points on (the “internal perimeter” of) γ_1 , and thus traps c between itself and (the “internal perimeter” of) γ_1 . The continuum loop consists of the excursion segment of γ_2 from A to B followed by γ_3 from B to A .

We note that in our inductive construction all loops are obtained essentially as in Fig. 1, except that the domain D_1 may itself be built out of more than one SLE_6 process. In the full plane version of our continuum nonsimple loop process, these SLE_6 processes (or the excursions into which they can be decomposed) are themselves parts of other constructed loops and it follows that every new loop touches previous ones (e.g., at the points A and B in Fig. 1). This leads to property (4) of Section 3, that any two of the continuum nonsimple loops are connected by a “path” of touching loops. The analogous lattice result concerns large percolation clusters that almost touch (i.e., their boundary contours are separated by only a *single* hexagonal cell) and can be explained in terms of standard “number of arms” arguments (see ref. 15 and Lemma 5 of ref. 20). It is also related to the high probability that “fjords” are of minimal width, a phenomenon observed numerically by Grossman–Aharony^(21,22) and explained by Aizenman–Duplantier–Aharony,⁽²³⁾ and which is a key ingredient of our scaling limit claim.

In addition to constructing the continuum loops from SLE_6 processes, one can also do the reverse—see property 5 of Section 3. We expect that this property, combined with some locality features in the spirit of those already known for SLE_6 ^(5, 13, 14) should be enough to characterize the full process of continuum nonsimple loops. Other characterizations of the full scaling limit, based on Cardy type crossing formulas,^(24, 25) have also been proposed—see, e.g., refs. 3–5, 27.

The basis for the scaling limit claim, presented in Section 4, is a construction for discrete site percolation on the triangular lattice \mathbb{T} , analogous to the construction of the process of continuum nonsimple loops. (We will generally think of the sites of the triangular lattice as the elementary cells of a regular hexagonal lattice \mathbb{H} embedded in the plane—see Fig. 2.) The argument that this discrete construction leads to a proof of the claimed limit is of course itself based on Schramm's⁽³⁾ and Smirnov's⁽⁴⁾ work on the scaling limit of the percolation “exploration process,” which we now briefly review.

Let D be a bounded simply connected open subset of the plane, with Jordan boundary ∂D (i.e., given by a closed continuous simple curve) and two distinct specified points a, b in ∂D . (Although the restriction to only Jordan regions can probably be dispensed with, it is convenient to have it, as we will throughout this paper.) There is a well-defined stochastic process $\gamma(t) = \gamma_{D, a, b}(t)$ for $t \in [0, \infty]$ in the closure \bar{D} with $\gamma(0) = a$, $\gamma(\infty) = b$ and Hölder continuous sample paths that is the trace of Schramm's chordal SLE_6 , the Stochastic Loewner Evolution with parameter $\kappa = 6$; this is conventionally defined first on the upper half plane with boundary points $0, \infty$ and then conformally mapped to D (see ref. 5).

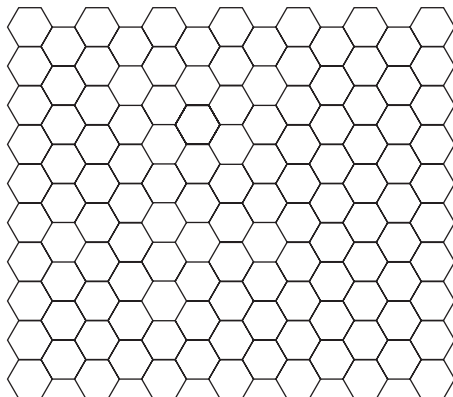


Fig. 2. Portion of the hexagonal lattice.

A major conclusion of Smirnov⁽⁴⁾ is that the scaling limit of a certain “exploration” process (see Section 4.2 for the definition) for critical independent site percolation on the triangular lattice (each site is equally likely to be yellow (minus) or blue (plus)) is the SLE_6 process γ . This is a statement about convergence in distribution, where the topology on sample paths is that of Aizenman–Burchard,⁽¹⁶⁾ which uses a supremum norm, but with monotonic reparametrizations of the paths allowed. The exploration process γ^δ runs along the edges of the hexagonal lattice that is dual to the triangular lattice of mesh size δ . It basically represents the contour separating blue clusters in D that reach the part $\partial_{a,b}D$ of the boundary ∂D that is traversed when touring ∂D counterclockwise from a to b and yellow clusters in D that reach the other part $\partial_{b,a}D$ of the boundary.

The sample paths of γ touch both ∂D and themselves many times, but they are noncrossing (and do not touch the same point more than twice or a boundary point even twice). The set $D \setminus \gamma[0, \infty]$ is a countable union of its connected components, which are open and simply connected—they are Jordan regions like the original region D , as will be discussed later in this section (see also refs. 11 and 14). If z is a deterministic point in D , then with probability one, z is not touched by $\gamma^{(26)}$ and so belongs to a unique one of these, that we denote $D_{a,b}(z)$.

There are four kinds of components which may be usefully thought of in terms of how a point z in the interior of the component was first “trapped” at some time t_1 by $\gamma[0, t_1]$ perhaps together with either $\partial_{a,b}D$ or $\partial_{b,a}D$: (1) those components whose boundary contains a segment of $\partial_{b,a}D$ between two successive visits at $\gamma_0(z) = \gamma(t_0)$ and $\gamma_1(z) = \gamma(t_1)$ to $\partial_{b,a}D$ (where here and below $t_0 < t_1$), (2) the analogous components with $\partial_{b,a}D$ replaced by the other part of the boundary $\partial_{a,b}D$, (3) those components formed when $\gamma_0(z) = \gamma(t_0) = \gamma(t_1) = \gamma_1(z)$ with γ winding about z in a counterclockwise direction between t_0 and t_1 , and finally (4) the analogous clockwise components.

The boundaries of these components, other than the segments of $\partial_{a,b}D$ or $\partial_{b,a}D$ in cases (1) and (2), are related to the “external perimeter” of chordal SLE_6 that was also studied by Smirnov,⁽⁴⁾ Lawler–Schramm–Werner,⁽¹¹⁾ and Werner.^(14, 27) Besides the exploration process itself, there are natural lattice analogues to these components, or more directly relevant for us, lattice analogues to their boundaries and to the points $\gamma_0(z)$, $\gamma_1(z)$ on their boundaries. We argue that it should follow from the work of Smirnov combined with other percolation arguments (see Section 4.4) that for any finitely (or countably) many deterministic points z_1, z_2, \dots in D , the joint distribution of the corresponding boundaries and points converges to the distribution of the continuum SLE_6 objects. This convergence also shows that the boundaries of these regions are Jordan curves (see ref. 23 and also refs. 11 and 14).

To obtain these new lattice analogues, one “fattens” the exploration process from being a path along the dual lattice (i.e., along the edges of $\delta\mathbb{H}$) to include all the blue and yellow sites that touch that path (i.e., the hexagons that have actually been explored while constructing the path γ^δ). Then one considers the connected (in the lattice sense) components of the difference between the set of all the sites in D and the set of all sites in the fattened exploration path. For more details, see Section 4.

2. CONSTRUCTION OF THE CONTINUUM NONSIMPLE LOOPS

In defining our process, we will freely switch between the real plane \mathbb{R}^2 , with x and y coordinates, and the complex plane \mathbb{C} , according to convenience. The basic ingredient in the algorithmic construction consists of a chordal SLE_6 path between two points a and b of the boundary ∂D of a given simply connected domain $D \subset \mathbb{C}$. The domains we will encounter in the construction are bounded open sets D whose boundaries ∂D are Jordan curves. This ensures that the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ can be mapped onto D via a conformal transformation that can be extended continuously to the boundary.

As we will explain soon, sometimes the two boundary points are “naturally” determined as a product of the construction itself, and sometimes they are given as an input to the construction. In the second case, there are various procedures which would yield the “correct” distribution for the resulting continuum nonsimple loop process; one possibility is as follows. Given a domain D , choose a and b so that, of all points in ∂D , they have maximal x -distance or maximal y -distance, whichever is greater. A crucial aspect of this procedure, as discussed in Section 4.4.2 below, is that there is a bounded away from zero probability that the resulting subdomains $D_{a,b}(z)$ have maximal x -distance (or else maximal y -distance) shrunk by a bounded away from one factor compared to the domain D . Another aspect, implicit in Sections 4.3 and 4.4, is that the corresponding (a, b) ’s of our discrete construction converge in distribution (in the scaling limit) to those of the continuum construction. This latter aspect and also the well-definedness of the above procedure would be resolved by showing that, in the context of our continuum construction, the choices of (a, b) are unique with probability one (for the starting domain, the unit disc \mathbb{U} , we take $(a, b) = (-i, i)$). We believe that this is so and proceed under that assumption, but in any case, the issue can be avoided by doing a randomized version of the above procedure in which (a, b) are chosen to be “fairly close” to having the maximal x -distance or y -distance.

To start our construction, we take the unit disc $\mathbb{U} = \mathbb{U}_1$ (later, to take a thermodynamic limit and extend the loop process to the entire plane, the

unit disc will be replaced by a growing sequence of large discs, \cup_R) and begin by “running” chordal SLE_6 inside \cup from $a = -i$ to $b = i$.

The resulting path $\gamma_{\cup, -i, i}$ (the trace of chordal SLE_6) touches itself and $\partial\cup$ (infinitely) many times. The set $\cup \setminus \gamma_{\cup, -i, i}[0, \infty]$ is a countable union of its connected components, which are open and simply connected. They can be of four different types, as explained in the introduction.

To conclude the first step (in this version of the construction), we consider all domains of type (1), corresponding to excursions of the SLE_6 path from the left portion of the boundary of the unit disc (the one from i to $-i$ counterclockwise). For each such domain D , the points a and b on its boundary are chosen to be respectively those points where the excursion ends and where it begins, that is, if $z \in D$, we set $a = \gamma_1(z)$ and $b = \gamma_0(z)$ (in the notation of Section 1). We then run chordal SLE_6 from a to b . The loop obtained by pasting together the excursion from b to a followed by the new SLE_6 path from a to b is one of our continuum loops. At the end of the first step, then, the procedure has generated countably many loops that touch the left side of the original boundary (the portion $\partial_{i, -i}\cup$ of the boundary of \cup); each of these loops touches the left side of the original boundary but may or may not touch the right side.

The last part of the first step also produces new domains, corresponding to the connected components of $D \setminus \gamma_{D, a, b}[0, \infty]$ for all domains D of type (1). Each one of these components, together with all the domains of type (2), (3), and (4) previously generated, is to be used in the next step of the construction, playing the role of the unit disc. For each one of these domains, we choose the “new a ” and “new b ” on the boundary as explained before, and then continue with the construction. Note that the “new a ” and “new b ” are chosen according to the rule explained at the beginning of this section also for domains of type (2), even though they are generated by excursions like the domains of type (1).

This iterative procedure produces at each step a countable set of loops. The limiting object, corresponding to the collection of all such loops, is our basic process. (Technically speaking, we should include also trivial loops fixed at each z in $\mathbb{C} \cup \{\infty\}$ so that the collection of loops is closed in an appropriate sense.⁽¹⁶⁾) Some of its properties will be given in the next section.

As explained, the construction carries on iteratively and can be performed simultaneously on all the domains that are generated at each step. We wish to emphasize, though, that the obvious monotonicity of the procedure, where at each step new paths are added independently in different domains, and new domains are formed from the existing ones, implies that any other choice of the order in which the domains are used would give the same result (i.e., produce the same limiting distribution), provided that

every domain that is formed during the construction is eventually used. In Section 4, when arguing that the lattice scaling limit coincides with our continuum nonsimple loop process, it will be convenient to utilize a different procedure in which each step involves only a single SLE_6 for a single domain. This will be done with the help of a deterministic set of points \mathcal{P} that are dense in \mathbb{C} and are endowed with a deterministic order. The domains will then be used one at a time, with domains containing higher ranked points of \mathcal{P} having a higher priority for order of being used.

In Section 4, arguments will be given as to why the process of loops we have just constructed is the scaling limit, as $\delta \rightarrow 0$, of the set of all cluster boundary contours for critical percolation on the portion of the triangular lattice of mesh size δ sitting within the disc \mathbb{U}_1 of radius 1, and with blue (plus) boundary conditions imposed. Of course, essentially the same construction and scaling limit results can be done on the disc \mathbb{U}_R of radius R . It is not hard to then verify that the limit in distribution of the loop process exists as $R \rightarrow \infty$ and that this represents the scaling limit of the set of all cluster boundary contours in the entire plane, with no boundary conditions needed. It is this process in the entire plane that we will consider in the next section of the paper dealing with properties of the loop process.

3. FEATURES OF THE CONTINUUM NONSIMPLE LOOP PROCESS

In this section we present a number of features that we argue will be valid for our process of continuum nonsimple loops in the plane. Some of them are direct consequences of the continuum algorithmic construction, while others become clear only in light of the analogous construction for discrete percolation, which will be presented in the next section.

The first feature is the scaling limit property—which is used to derive the other properties. A sketch of the derivation of the scaling limit and other properties is given in the next section of the paper. The scaling limit property (1) is a distributional statement; properties (2)–(4) all involve statements that are valid with probability one; property (5) is a bit of a hybrid.

(1) The continuum nonsimple loop process is the scaling limit of the set of all boundary contours for critical site percolation on the triangular lattice.

(2) This process is a random collection of noncrossing, continuous loops on the plane. The loops touch themselves and each other many times, but no three or more loops can come together at the same point, and a single loop cannot touch the same point more than twice.

(3) Any deterministic point of the plane is surrounded by an infinite family of nested loops with diameters going to both zero and infinity; any annulus about that point with inner radius $r_1 > 0$ and outer radius $r_2 < \infty$ contains only a finite number of loops. Consequently, any two distinct deterministic points of the plane are separated by loops winding around them.

(4) Any two loops are connected by a finite “path” of touching loops.

(5) For a (deterministic) Jordan region D with two boundary points a and b , a process distributed as chordal SLE_6 from a to b can be constructed starting from the continuum nonsimple loops (in the whole plane) by doing a continuum analogue of what is done on the lattice to piece together cluster boundary segments to give the lattice percolation exploration process (see below).

We conclude this section of the paper with a more detailed explanation of the construction just mentioned in property (5). To do the construction, it is useful to first convert all the loops into directed ones. There is one binary choice to be made: any one loop can be given either the clockwise or counterclockwise direction and then all other loops are automatically determined (via the natural nesting tree structure of the set of all the loops) by requiring that the set of all loops (ordered by nesting) about any deterministic point alternate in direction. Back on the percolation lattice the two choices correspond to either having yellow just to the left of the directed path and blue just to the right or vice versa; the two choices are also of course related by a global color flip.

For convenience, let us suppose that a is at the bottom and b is at the top of D so that the boundary is divided into a left and right part by these two points. The desired path from a to b is then put together using *all* of the following directed segments of the loops (most of the analogous types of segments for the lattice exploration process may be seen in Fig. 3): (i) “excursions” from the left part to the right part of the boundary (they touch each of the two boundary parts at exactly one point), (ii) “excursions” from right to left, (iii) excursions from the left part to itself (they touch that part of the boundary at exactly two points) which do not touch the right part and which are maximal in that there is not another such excursion between them and the right part, and (iv) the analogous excursions from the right part to itself.

There are countably many excursions of types (i) and (ii) which are ordered from lower to higher and alternate between types (i) and (ii). The segment of the right boundary between where an excursion of type (i) ends

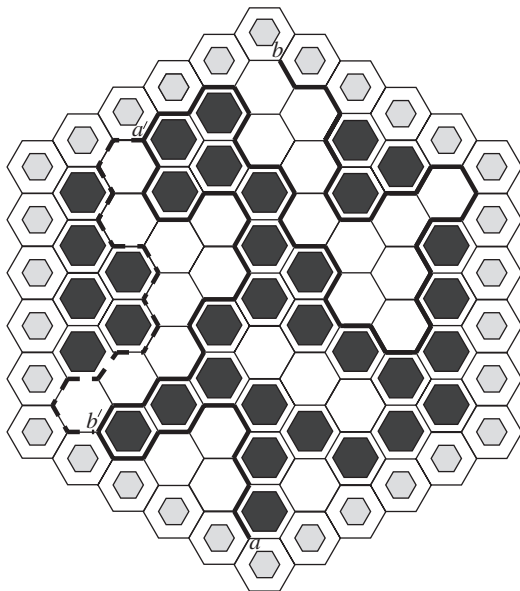


Fig. 3. Construction of the outer contour of a cluster of yellow/minus (white in the figure) hexagons in two steps by means of a first exploration from the vertex a to b (heavy line), followed by a second one from a' to b' (heavy broken line). The outer layer of hexagons does not belong to the domain where the explorations are carried out, but represents its monochromatic blue/plus external site boundary.

and the next excursion of type (ii) begins supports countably many excursions of type (iv) which are also ordered from lower to higher. These may be all pieced together (they don't quite touch so a limit is needed) in order that a continuous path connecting the type (i) to the type (ii) excursion is obtained. Using such connecting paths on the right and the analogous paths on the left that connect the end of a type (ii) to the beginning of the next type (i), one can connect all the type (i) and (ii) excursions in order and obtain finally the desired path from a to b .

4. THE CONTINUUM NONSIMPLE LOOP PROCESS AS SCALING LIMIT

In this section we will introduce a discrete inductive construction which is analogous to the continuum construction given in Section 2. Our interest in the discrete construction comes from the claim that the continuum one is its scaling limit. This requires comparing the two

constructions. In order to do so, we first reorganize the continuum one and introduce some notation.

4.1. Priority-Ordered Continuum Construction

We want to arrange the continuum construction in such a way that each step corresponds to a single new SLE_6 path. To do that, we need to order the domains present at the beginning of each stage (which is the term we use for a group of successive single steps), so as to choose which ones to use in the steps of that stage. The domains are the connected components that the original domain is broken up into by *all* the SLE_6 paths constructed up to the beginning of the new stage. The ordering will be done with the help of the deterministic ordered set of points \mathcal{P} , dense in \mathbb{C} , introduced in Section 2.

The first step and stage consists of an SLE_6 path $\gamma_1 = \gamma_{\cup, -i, i}$ inside \cup from $-i$ to i which, as explained in Section 2, produces many domains which are the connected components of the set $\cup \setminus \gamma_1[0, \infty]$. These domains can be priority-ordered using points in \mathcal{P} , according to the rank of the highest ranking point of \mathcal{P} that each contains. The priority orders of domains change as the construction proceeds.

The second stage of the construction consists of two SLE_6 paths, γ_2 and γ_3 , that are produced in the two domains with highest priority at the end of the first stage, the priority being determined using the points of \mathcal{P} and the starting and ending points for domains that are not of type (1) being chosen as explained in Section 2.

In general, for the k th stage of the construction, k SLE_6 paths are produced in those k domains present at the end of the last stage with highest priority, again using the points of \mathcal{P} for ranking the domains. This way of organizing the construction does not affect the final result, as discussed in Section 2, and has the advantage that to each step corresponds a single SLE_6 path, with the SLE_6 paths ordered.

4.2. Discrete Exploration and Loop Construction

We will organize the discrete construction, which we will present soon, in the same way. Before doing that, though, we briefly introduce its key ingredient—the discrete exploration process for a general simply connected set D^δ of hexagons.

To begin, we denote by ∂D^δ the edge boundary. For two points, a, b in ∂D^δ suitably chosen at the vertices of two hexagons, the usual exploration process⁽⁴⁾ (see also refs. 13, 14, and 27) with \pm boundary conditions (i.e., blue hexagons just outside the counterclockwise portion $\partial_{a,b} D^\delta$ of ∂D^δ from

a to b and yellow hexagons just outside the other portion $\partial_{b,a}D^\delta$) can be described as a sort of self-avoiding random walk on the edges of the hexagons contained in D^δ that moves left (with respect to the current direction of exploration) when a blue hexagon is encountered and right when a yellow one is encountered.

We use this rule for \pm boundary conditions, and *also* for $+$ (blue) boundary conditions, proceeding at the boundary *as if* we had \pm boundary conditions (see Fig. 3). For \mp and $-$ (yellow) boundary conditions, we use the “opposite” (with respect to color) rule. We remark that although the exploration process itself changes under a color flip of the boundary conditions, its distribution is color-blind.

The interpretation of the exploration process depends on whether the boundary condition is monochromatic or not. Let ΔD^δ be the external (outer) site boundary of D^δ , with $\Delta_{a,b}D^\delta$ and $\Delta_{b,a}D^\delta$ representing the portions next to $\partial_{a,b}D^\delta$ and $\partial_{b,a}D^\delta$ respectively.

- For regions with \pm (respectively, \mp) boundary conditions, the exploration path represents the contour separating the blue (respectively, yellow) cluster that contains $\Delta_{a,b}D^\delta$ from the yellow (respectively, blue) cluster that contains $\Delta_{b,a}D^\delta$.

- For regions with monochromatic blue (respectively, yellow) boundary conditions, the exploration path represents portions of the outer boundary contours of yellow (respectively, blue) clusters touching $\partial_{b,a}D^\delta$ and adjacent to blue (respectively, yellow) hexagons that are the starting point of a blue (respectively, yellow) path (possibly an empty path) that reaches $\partial_{a,b}D^\delta$, pasted together using portions of $\partial_{b,a}D^\delta$.

Next, we show how to get the complete outer contour of a monochromatic (say, yellow) cluster by twice using the exploration process described above (see Fig. 3). Consider a large simply connected domain D^δ surrounded by blue hexagons, which we can identify with ΔD^δ . D^δ will contain many clusters of both colors in its interior. We pick two suitably chosen points $a, b \in \partial D^\delta$ and perform the exploration from a to b .

While performing the exploration process, we discover the color of the hexagons that touch the exploration path. We want to keep track of that information. As a result, at the end of the exploration process we have three “paths:” the exploration path γ^δ along the edges of the hexagonal lattice, and respectively the “paths” Γ_Y^δ and Γ_B^δ along the (respectively, yellow or blue) sites of the triangular lattice that touch it (i.e., those hexagons that have at least one edge belonging to the exploration path). The latter lattice “paths” are not in general simple, as they can form loops and have dangling ends.

The set $D^\delta \setminus \{\Gamma_Y^\delta \cup \Gamma_B^\delta\}$ is the union of its connected components (in the lattice sense), which are simply connected. There are four types of components which may be usefully thought of in terms of their external site boundaries: (1) those components whose site boundary contains both sites in Γ_Y^δ and $\Delta_{b,a}D^\delta$, (2) the analogous components with $\Delta_{b,a}D^\delta$ replaced by $\Delta_{a,b}D^\delta$ and Γ_Y^δ by Γ_B^δ , (3) those components whose site boundary only contains sites in Γ_Y^δ , and finally (4) the analogous components with Γ_Y^δ replaced by Γ_B^δ .

If we now take a region of type (1), there are natural starting and ending points (where the excursion that produces that region respectively ends and starts; e.g., a', b' in Fig. 3) for an exploration process within it. Performing such an exploration process inside the specified domain of type (1) and pasting the new exploration path together with the portion of a previous exploration path corresponding to the excursion that produced that domain of type (1) will generate a loop along the edges of the hexagonal lattice. The loop is the outer contour of a yellow cluster that touches $\partial_{b,a}D^\delta$ and is adjacent (on its “right”) to blue hexagons, each of which is the starting point of a blue path to $\partial_{a,b}D^\delta$.

Analogous exploration processes in the other regions of type (1) produce similar loops on the edges of $\delta\mathbb{H}$ that are also boundary contours. In fact, every domain with \pm (or \mp) boundary conditions obtained during the discrete algorithmic construction that we are about to present will contain an exploration path which, pasted together with the appropriate part of a previous exploration path, provides the complete outer boundary contour of a monochromatic cluster.

4.3. Full Discrete Construction

We now give the algorithmic construction for discrete percolation which is the analogue of the continuum one. Each step of the construction is a single percolation exploration process; the order of successive steps is organized as in the continuum construction detailed at the beginning of this section. We start with the set D_0^δ of hexagons that are contained in the unit disc \mathbb{U} and will make use of the deterministic countable ordered set \mathcal{P} of points dense in \mathbb{C} that was introduced in Section 2.

The first step consists of an exploration process inside D_0^δ . For this, we need to select two points a and b in ∂D_0^δ . We choose for a some vertex close to $-i$, and for b one close to i . The first exploration produces a path γ_1^δ and, for δ small, many new domains of all four types. These domains are ordered with the help of points in \mathcal{P} as in the continuum case, and that order is used, at each stage of the process, to determine the next group of exploration processes. So, for the second stage of the construction, two

domains are chosen and explored, and so on. With this choice, the exploration processes and paths are naturally ordered: $\gamma_1^\delta, \gamma_2^\delta, \dots$.

Each exploration process of course requires choosing a starting and ending point, which is done mimicking what is done in the continuum case (with some adjustments due to the discrete nature of the lattice). For domains of type (1), with \pm or \mp boundary conditions, the choice is the natural one, explained before, which produces a loop using the edges of $\delta\mathbb{H}$. For a domain D_k^δ (used at the k th step) of type other than (1), and therefore with monochromatic boundary conditions, two vertices are chosen that are close to the two points of ∂D_k^δ selected according to the rule given in Section 2.

The procedure continues iteratively, with regions of type (2), (3), and (4), which have monochromatic boundaries, playing the role played in the first step by D_0^δ . As the construction continues, new loops along the edges of the hexagonal lattice are formed which correspond to the outer boundary contours of constant sign (monochromatic) clusters.

4.4. Ingredients for Convergence

By comparing the discrete and continuum version of the algorithmic construction, and using repeated applications of Smirnov's work,⁽⁴⁾ we will argue that for any fixed k , the first k steps of the discrete construction converge (jointly, in distribution) to the first k steps of the continuum construction, as $\delta \rightarrow 0$. This claim is an extension of the discussion near the end of Section 1 about convergence in distribution of certain lattice boundaries and points to their continuum analogues. We note that one complication is due to the fact that the boundaries of the domains where the exploration processes are performed are not deterministic, but are themselves obtained using exploration processes. Some continuity arguments are therefore needed.

4.4.1. Matching Continuous and Discrete Domains and Loops

A key ingredient is the observation that the probability of "fjords" of width larger than the minimal one goes to zero in the scaling limit.⁽²³⁾ This ensures that the domains and loops generated at various steps of the continuum construction are the limits of corresponding domains and loops produced in the discrete one, so that, e.g., one can identify, with probability going to one as $\delta \rightarrow 0$, the domain containing a point c at a given step of the continuum construction with the domain containing c at the equivalent step of the discrete one.

4.4.2. Finding Large Contours in $O(1)$ Steps

The discrete algorithm will reach and discover all the boundary contours inside \mathbb{U} ; moreover we argue that the number of steps $K_\varepsilon(\delta)$ needed for the discrete algorithm to recover all contours in \mathbb{U} of diameter larger than a given $\varepsilon > 0$ is bounded (in probability) as $\delta \rightarrow 0$.

This uses the observation that the discrete algorithm cannot “skip” a contour and move to explore the domain inside it and the fact that the maximum diameter of the domains present inside \mathbb{U} after k steps of the discrete algorithm tends to zero in probability as $k \rightarrow \infty$, $\delta \rightarrow 0$. To understand the last fact, first of all notice that the construction cannot produce “too many” distinct domains of diameter greater than ε , or else there would be too many disjoint “macroscopic” monochromatic paths (the site boundaries of those domains) in $\delta\mathbb{T} \cap \mathbb{U}$ to satisfy the multiple crossing probability bounds of ref. 16. Consider now a domain D^δ with points a and b on the boundary ∂D^δ chosen because they have, among all points in ∂D^δ , maximal x -distance. Then standard percolation arguments^(29,30) ensure that, with bounded away from zero probability, the maximal x -distance between points on the boundary of each of the components that D^δ is split up into by effect of the exploration process is smaller than, say, two thirds of the x -distance between a and b . (Notice also that each newly formed domain is “unexplored territory” on which no information is available before the exploration process inside it begins.)

The proof of property (1) of Section 3 is completed by first letting $\varepsilon \rightarrow 0$ and then by taking the thermodynamic limit (to obtain a loop process in the entire plane, as discussed at the end of Section 2).

4.5. Properties of the Continuum Loop Process

We now turn to brief sketches of the derivations of the other properties presented in Section 3.

(2) The noncrossing property of contours is preserved in the scaling limit, and the fact that they touch themselves and each other follows fairly directly from the continuum construction (see the discussion below about property (4)). The properties that no three or more loops can come together at the same point and a single loop cannot touch the same point more than twice follow from standard “number of arms” arguments (see ref. 15 and Lemma 5 of ref. 20).

(3) Both the fact that any deterministic point of the plane is surrounded by infinitely many loops and the claim about the inner and outer radii of loops surrounding a given point follow from property (1) combined with standard percolation arguments^(29,30) (see also Lemma 3 of ref. 20).

(4) This property follows fairly directly from the continuum construction, as discussed in Section 1. As explained in the introduction, the analogous lattice result concerns large clusters of the same sign that almost touch and the existence of “macroscopic fjords” only of minimal width (see refs. 21–23). For example, the existence of a long double monochromatic layer of hexagons separating two large clusters of the same color would give rise to six disjoint “macroscopic” paths of hexagons not all of the same color which start within a “microscopic” distance of each other. The probability of this happening goes to zero as $\delta \rightarrow 0$.

(5) This property is proved by noting that the usual lattice exploration process can be realized as a discrete version of the continuum exploration procedure outlined at the end of Section 3. By ref. 4, it is enough to show that the lattice version converges to the continuum one.

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